Int. J. Solids Structures Vol. 19, No. 10, pp. 873–883, 1983
Printed in Great Britain.
0020-7683/83 $3.00 + .00

ISOTROPIC STOCHASTIC VISCO-ELASTIC STRAIN MODELLED AS A SECOND MOMENT WHITE NOISE FIELD

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(*Received* 10 *May* 1982; *in revised form* 1 *February* 1983)

Abstract—Constitutive equations for statistically isotropic triaxial visco-elastic stochastic creep are formulated by generalization from a previous formulated theory of uniaxial stochastic creep. The constitutive equations define a random strain tensor as function of a given deterministic stress tensor history. Compatibility constraints imply that also stresses become random even for a deterministic external load history. In order to reach solutions in terms of means and covariances it turns out to be necessary to linearize the constitutive equations with respect to the stress tensor increments. This Part 1 of the paper terminates by discussing some solutions of relevance for comparisons with the uniaxial theory of combined elongation end bending of slender prismatic bodies. Part 2 discusses solutions for biaxial stress histories that are homogeneous in the mean.

INTRODUCTION

The purpose of this paper is to extend to three dimensions of uniaxial second moment white noise creep model formulated by the writer[3]. The model is restricted to deal with *isotropic* materials and it is supposed to be an extension of usual linear visco-elasticity theory to include stochastic behavior.

In this Part 1 of the paper the stochastic constitutive equations are formulated and they are exempified by the special case of Poisson process viscous creep. Due to the stochastic nature of the constitutive equations also the equilibrium equations and the compatibility equations become stochastic. In order to reach even approximate solutions in terms of means and covariances to this set of stochastic equations and the boundary conditions, it is necessary to linearize the constitutive equations with respect to the random stress field. With this approximation introduced particularly simple solutions are possible for the average strain in a body subjected to given external forces. In this connection the theory of bending of beams given in [3] is discussed in relation to the triaxial theory presented herein.

Part 2 of the paper considers biaxial stress fields and presents solutions for the covariance structure in examples with stress fields that in the mean are homogeneous.

CONSTITUTIVE EQUATIONS

Let $\mathbf{r} = (x, y, z)$ denote the generic point of the considered material body $\mathcal{B}$, and let t and τ denote time. The creep strain tensor $\epsilon_{ij}(\mathbf{r}, t)$ resulting from an imposed stress tensor history $\bar{\sigma}(\mathbf{r}, \tau)$, $0 \le \tau \le t$, is modeled by the stochastic integral

$$\epsilon_{ij}(\mathbf{r}, t) = \int_{t=0}^{t} \int_{\bar{u}=\bar{\sigma}(\mathbf{r},\tau)}^{\bar{\sigma}(\mathbf{r},\tau)+d\bar{\sigma}(\mathbf{r},\tau)} S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})\, du_{rs} \tag{1}$$

using the summation convention of tensor algebra. The stress tensor increment at time τ is denoted $d\bar{\sigma}(\mathbf{r}, \tau)$. For fixed t the integrand $S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})$ is a second moment white noise random tensor process with its parameter set defined by the generic point $(\mathbf{r}, \tau, \bar{u})$, i.e. its parameter set is the cartesian product $\mathcal{B} \times R_0 \times \{\text{space of stress tensors } \bar{u}\}$.

The white noise property of the integrand necessitates the particular way of writing the stochastic integral of eqn (1). The integration with respect to the stress tensor is curve integration along a straight line joining the points $\bar{\sigma}(\mathbf{r}, \tau)$ and $\bar{\sigma}(\mathbf{r}, \tau) + d\bar{\sigma}(\mathbf{r}, \tau)$ in the stress tensor space. Even with the assumption that the white noise tensor process S_{ijrs} possesses sample functions that almost surely are continuous in the stress tensor $\bar{u}$, it will not be suitable to apply the mean value theorem of integral theory in order to eliminate the inner integral of

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eqn (1). The result of such an elimination would be the equation

$$\epsilon_{ij}(\mathbf{r}, t) = \int_{\tau=0}^{t} S_{ijrs}(\mathbf{r}, t, \tau, \bar{\sigma}(\mathbf{r}, \tau) + \theta\, \mathrm{d}\bar{\sigma}(\mathbf{r}, \tau))\, \mathrm{d}\sigma_{rs}(\mathbf{r}, \tau) \tag{1a}$$

in which θ is some random variable with outcomes in the interval [0, 1]. There seems to be no obvious way to come from eqn (1a) to the covariance structure of the strain tensor field when given the mean and covariance properties of S_{ijrs}. This difficulty disappears for the formulation given by eqn (1). The fundamental importance of not cancelling the dependence on $\bar{\sigma}(\mathbf{r}, \tau) + \theta\, \mathrm{d}\bar{\sigma}(\mathbf{r}, \tau)$ in eqn (1a) is explained in the Remark in [3]. It is emphasized that except for this discussion related to eqn (1a) there will be no assumption herein on continuity of the sample functions of S_{ijrs} because it suffices to let the stochastic integral of eqn (1) be in the mean square sense. The second moment white noise tensor S_{ijrs} has the following properties. The mean

$$E[S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})] = K_{ijrs}(t, \tau) \tag{2}$$

is an isotropic tensor which is a function solely of present time t and time τ of stress increment application. The covariance is

$$\begin{aligned} &\mathrm{Cov}\,[S_{ijrs}(\mathbf{r}_1, t_1, \tau_1, \bar{u}_1),\ S_{klpq}(\mathbf{r}_2, t_2, \tau_2, \bar{u}_2)] \\ &= \delta(\mathbf{r}_2 - \mathbf{r}_1)\delta(\tau_2 - \tau_1)c_{ijklrspq}(\bar{u}_2 - \bar{u}_1, t_1, t_2, \tau_1) \end{aligned} \tag{3}$$

in which $c_{ijklrspq}(\bar{u}_2 - \bar{u}_1, t_1, t_2, \tau_1)$ is an isotropic tensor which besides being a function of t_1, t_2, τ_1 is a function of the stress tensor increment $\bar{u}_2 - \bar{u}_1$. With reference to the assumed isotropy of the material the dependency of the increment $\bar{u}_2 - \bar{u}_1$ is required to be form-invariant with respect to cartesian coordinate transformations. This implies that the functional dependency of $\mathrm{d}\bar{u} = \bar{u}_2 - \bar{u}_1$ must be solely through the invariants of $\mathrm{d}\bar{u}$. In the following theory it is assumed that only the norm

$$\|\mathrm{d}\bar{u}\| = \sqrt{(\mathrm{d}u_{ij}\, \mathrm{d}u_{ij})} \tag{4}$$

and no other invariants of $\mathrm{d}\bar{u}$ matter.

In eqn (3) the symbol $\delta(\cdot)$ is the usual Dirac delta function and $\delta(\mathbf{r}_2 - \mathbf{r}_1) = \delta(x_2 - x_1)\,\delta(y_2 - y_1)\,\delta(z_2 - z_1)$.

The mean strain at $\mathbf{r}$ to time t becomes

$$\begin{aligned} E[\epsilon_{ij}(\mathbf{r}, t)] &= \int_{\tau=0}^{t} \int_{\bar{\sigma}(\mathbf{r}, \tau)}^{\bar{\sigma}(\mathbf{r}, \tau)+\mathrm{d}\bar{\sigma}(\mathbf{r}, \tau)} E[S_{ijrs}(\mathbf{r}, t, \tau, \bar{u})]\, \mathrm{d}u_{rs} \\ &= \int_{\tau=0}^{t} K_{ijrs}(t, \tau)\, \mathrm{d}\sigma_{rs}(\mathbf{r}, \tau). \end{aligned} \tag{5}$$

Since $K_{ijrs}(t, \tau)$ is an isotropic tensor which is symmetric in both i and j and in r and s, and since $\mathrm{d}\sigma_{rs}$ is symmetric, the sufficiently most general form of $K_{ijrs}(t, \tau)$ is, [4: p. 98],

$$K_{ijrs}(t, \tau) = A(t, \tau)\delta_{ij}\delta_{rs} + B(t, \tau)\delta_{ir}\delta_{js} \tag{6}$$

in which $A(t, \tau)$ and $B(t, \tau)$ are scalar functions of t and τ, and δ_{ij} is Kronecker's delta. By introducing a Poisson ratio function

$$\nu(t, \tau) = -\frac{A(t, \tau)}{A(t, \tau) + B(t, \tau)} \tag{7}$$

and a creep function

$$C(t, \tau) = A(t, \tau) + B(t, \tau) \tag{8}$$

the mean may consequently be written as

$$E[\epsilon_{ij}(\mathbf{r}, t)=\int_{\tau=0}^{t}(1+\nu(t,\tau))C(t,\tau)\,\mathrm{d}\sigma_{ij}(\mathbf{r},\tau)-\delta_{ij}\int_{\tau=0}^{t}\nu(t,\tau)C(t,\tau)\,\mathrm{d}\sigma_{ss}(\mathbf{r},\tau). \tag{9}$$

This is recognized as the usual constitutive equation for isotropic linear visco-elasticity except that the left hand side is the mean strain tensor.

For the covariance we get

$$\begin{aligned}&\mathrm{Cov}\,[\epsilon_{ij}(\mathbf{r}_1,t_1),\epsilon_{kl}(\mathbf{r}_2,t_2)]\\&=\mathrm{Cov}\left[\int_0^{t_1}\int_{\bar\sigma_1}^{\bar\sigma_1+\mathrm{d}\bar\sigma_1}S_{ijrs}(\mathbf{r}_1,t_1,\tau_1,\bar u_1)\,\mathrm{d}u_{rs},\int_0^{t_2}\int_{\bar\sigma_2}^{\bar\sigma_2+\mathrm{d}\bar\sigma_2}S_{klpq}(\mathbf{r}_2,t_2,\tau_2,\bar u_2)\,\mathrm{d}u_{pq}\right]\\&=\delta(\mathbf{r}_2-\mathbf{r}_1)\int_{\tau_1=0}^{t_1}\int_{\tau_2=0}^{t_2}\delta(\tau_2-\tau_1)\int_{\bar u_1=\bar\sigma_1}^{\bar\sigma_1+\mathrm{d}\bar\sigma_1}\mathrm{d}u_{rs}\int_{\bar u_2=\bar\sigma_2}^{\bar\sigma_2+\mathrm{d}\bar\sigma_2}c_{ijklrspq}\,(\bar u_2-\bar u_1,t_1,t_2,\tau_1)\,\mathrm{d}u_{pq}\end{aligned} \tag{10}$$

in which $\bar\sigma_1=\bar\sigma(\mathbf{r}_1,\tau_1)$ and $\bar\sigma_2=\bar\sigma(\mathbf{r}_2,\tau_2)$. Due to the delta function factors $\delta(\mathbf{r}_2-\mathbf{r}_1)\ \delta(\tau_2-\tau_1)$ the stress tensors $\bar\sigma_2$ and $\mathrm{d}\bar\sigma_2$ may be changed to $\bar\sigma_1$ and $\mathrm{d}\bar\sigma_1$ in the last integral. By the substitution $\bar u_1=u\,\mathrm{d}\bar\sigma_1$, $\bar u_2=v\,\mathrm{d}\bar\sigma_1$ the inner double integral thus get the value

$$\begin{aligned}&\int_{u=0}^{1}\int_{v=0}^{1}c_{ijklrspq}((v-u)\,\mathrm{d}\bar\sigma_1,t_1,t_2,\tau_1)\,\mathrm{d}(v\,\mathrm{d}\sigma_{rs})\,\mathrm{d}(u\,\mathrm{d}\sigma_{pq})\\&=\mathrm{d}\sigma_{rs}\,\mathrm{d}\sigma_{pq}\left(\int_0^{1/2}\mathrm{d}z\int_{-z}^{z}+\int_{1/2}^{1}\mathrm{d}z\int_{-(1-z)}^{1-z}\right)c_{ijklrspq}(2y\,\mathrm{d}\bar\sigma_1,t_1,t_2,\tau_1)\,\mathrm{d}y\end{aligned} \tag{11}$$

where the last integral follows by the substitution $2y=v-u$, $2z=v+u$. By the second moment white noise property of S_{ijrs} the integrand of eqn (11) as a function of y behaves like a Dirac delta function. Furthermore, as mentioned above, the functional dependency of $2y\,\mathrm{d}\bar\sigma_1$ is solely through the norm $2|y|\,\|\mathrm{d}\bar\sigma_1\|$. Thus the integral of eqn (11) has the form

$$\mathrm{d}_{ijklrspq}(t_1,t_2,\tau_1)\frac{\mathrm{d}\sigma_{rs}\,\mathrm{d}\sigma_{pq}}{\|\mathrm{d}\bar\sigma\|} \tag{12}$$

where $\|\mathrm{d}\bar\sigma\|=\sqrt{(\mathrm{d}\sigma_{ij}\,\mathrm{d}\sigma_{ij})}$, and the function $\mathrm{d}_{ijklrspq}(t_1,t_2,\tau_1)$ like $c_{ijklrspq}$ is an isotropic tensor of eighth order. By substitution into eqn (10) the covariance becomes

$$\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1,t_1),\epsilon_{kl}(\mathbf{r}_2,t_2)]=\delta(\mathbf{r}_2-\mathbf{r}_1)\int_0^{t_1}\mathrm{d}_{ijklrspq}(t_1,t_2,\tau)\frac{\mathrm{d}\sigma_{rs}(\mathbf{r}_1,\tau)\,\mathrm{d}\sigma_{pq}(\mathbf{r}_1,\tau)}{\|\mathrm{d}\bar\sigma(\mathbf{r}_1,\tau)\|} \tag{13}$$

valid for $t_1\le t_2$.

From the general theory of isotropic tensors, [5: p. 260], it is known that an isotropic tensor of 8th order may be written as a linear combination of products

$$\delta_{ij}\delta_{kl}\delta_{rs}\delta_{pq} \tag{14}$$

of 4 Kronecker's deltas. These products are generated by letting the string of indices *ijklrspq* run through all 8! permutations. However, due to the symmetry of Kronecker's delta and the irrelevance of the order of the factors there are only $8!/(2^4\,4!)=105$ different products. Furthermore, since 4 of the indices in eqn (13) are summation indices and since Kronecker's delta acts as an index substitution operator, only the following 15 tensors make up the linear combination of the integrand of eqn (13):

$$
\begin{array}{l}
d\sigma_{ij}\, d\sigma_{kl} \\
d\sigma_{ik}\, d\sigma_{jl} \\
d\sigma_{kj}\, d\sigma_{il} \\
\\
\delta_{ij}\, d\sigma_{kl}\, d\sigma_{ss} \\
\delta_{kl}\, d\sigma_{ij}\, d\sigma_{ss} \\
\delta_{jl}\, d\sigma_{ik}\, d\sigma_{ss} \\
\delta_{jk}\, d\sigma_{il}\, d\sigma_{ss} \\
\delta_{il}\, d\sigma_{jk}\, d\sigma_{ss} \\
\delta_{ik}\, d\sigma_{jl}\, d\sigma_{ss} \\
\\
\delta_{ij}\delta_{kl}\, (d\sigma_{ss})^2 \\
\delta_{ik}\delta_{jl}\, (d\sigma_{ss})^2 \\
\delta_{il}\delta_{kj}\, (d\sigma_{ss})^2 \\
\delta_{ij}\delta_{kl}\, d\sigma_{rs}\, d\sigma_{rs} \\
\delta_{ik}\delta_{jl}\, d\sigma_{rs}\, d\sigma_{rs} \\
\delta_{il}\delta_{kj}\, d\sigma_{rs}\, d\sigma_{rs}
\end{array}
\tag{15}
$$

which all should be divided by the scalar $\|d\bar{\sigma}\|$. A further reduction of the possibilities is obtained by noting that there should be symmetry with respect to i, j and with respect to k, l. Denoting the scalar coefficient functions by a_1 through a_{15} in the order of the list of tensors given above this symmetry requires that $a_2 = a_3$, $a_6 = a_7 = a_8 = a_9$, $a_{11} = a_{12}$, $a_{14} = a_{15}$. Since for $t_1 \le t_2$ we have $\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}, t_1), \epsilon_{kl}(\mathbf{r}, t_2)] = \mathrm{Cov}[\epsilon_{ij}(\mathbf{r}, t_1), \epsilon_{kl}(\mathbf{r}, t_1)] = \mathrm{Cov}[\epsilon_{kl}(\mathbf{r}, t_1), \epsilon_{ij}(\mathbf{r}, t_1)] = \mathrm{Cov}[\epsilon_{kl}(\mathbf{r}, t_1), \epsilon_{ij}(\mathbf{r}, t_2)]$, it follows that also $a_4 = a_5$. Thus the covariance of eqn (13) may be written

$$
\begin{aligned}
\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1), \epsilon_{kl}(\mathbf{r}_2, t_2)] = \delta(\mathbf{r}_2 - \mathbf{r}_1) \int_{\tau=0}^{t_1} [& b_1\, d\sigma_{ij}\, d\sigma_{kl} + b_2(d\sigma_{ik}\, d\sigma_{jl} + d\sigma_{kj}\, d\sigma_{il}) \\
& + b_3(\delta_{ij}\, d\sigma_{kl} + \delta_{kk}\, d\sigma_{ij})\, d\sigma_{ss} \\
& + b_4(\delta_{jl}\, d\sigma_{ik} + \delta_{jk}\, d\sigma_{il} + \delta_{il}\, d\sigma_{jk} + \delta_{ik}\, d\sigma_{jl})\, d\sigma_{ss} \\
& + \delta_{ij}\delta_{kl}(b_5(d\sigma_{ss})^2 + b_6\, d\sigma_{rs}\, d\sigma_{rs}) \\
& + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})(b_7(d\sigma_{ss})^2 + b_8\, d\sigma_{rs}\, d\sigma_{rs})] \frac{1}{\|d\bar{\sigma}\|}
\end{aligned}
\tag{16}
$$

for $t_1 \le t_2$. Herein $b_1, \ldots, b_8$ are scalar functions of t_1, t_2, τ while the stress tensor increment $d\bar{\sigma}$ is taken at $(\mathbf{r}_1, \tau)$.

In order to be able to set up models that determine the different scalar functions in connection with some simple stress states, it is necessary to write the integrand of eqn (16) out in its components. It suffices, however, to do this in the coordinate system of principal stress increments, i.e. to assume that $d\sigma_{ij} = 0$ for $i \ne j$. For the present purpose we therefore assume that the coordinate system of principal stress axes is constant in time. We get

$$
\begin{aligned}
\mathrm{Cov}[\epsilon_{11}(\mathbf{r}_1, t_1), \epsilon_{11}(\mathbf{r}_2, t_2)] = \delta(\mathbf{r}_2 - \mathbf{r}_1) \int_{\tau=0}^{t_1} [& (b_1 + 2b_2)\, d\sigma_{11}^2 \\
& + (2b_3 + 4b_4)\, d\sigma_{11}(d\sigma_{11} + d\sigma_{22} + d\sigma_{33}) + (b_5 + 2b_7)(d\sigma_{11} + d\sigma_{22} + d\sigma_{33})^2 \\
& + (b_6 + 2b_8)(d\sigma_{11}^2 + d\sigma_{22}^2 + d\sigma_{33}^2)] \frac{1}{\|d\bar{\sigma}\|}
\end{aligned}
\tag{17}
$$

and correspondingly for ϵ_{22} and ϵ_{33}.

$$
\begin{aligned}
\mathrm{Cov}[\epsilon_{12}(\mathbf{r}_1, t_1), \epsilon_{12}(\mathbf{r}_2, t_2)] = \delta(\mathbf{r}_2 - \mathbf{r}_1) \int_{\tau=0}^{t_1} [& b_2\, d\sigma_{11}\, d\sigma_{22} + b_4(d\sigma_{11} + d\sigma_{22})(d\sigma_{11} + d\sigma_{22} + d\sigma_{33}) \\
& + b_7(d\sigma_{11} + d\sigma_{22} + d\sigma_{33})^2 + b_8(d\sigma_{11}^2 + d\sigma_{22}^2 + d\sigma_{33}^2)] \frac{1}{\|d\bar{\sigma}\|}
\end{aligned}
\tag{18}
$$

and correspondingly for ϵ_{13} and ϵ_{23}.

$$\mathrm{Cov}\,[\epsilon_{11}(\mathbf{r}_1, t_1), \epsilon_{22}(\mathbf{r}_2, t_2)] = \delta(\mathbf{r}_2 - \mathbf{r}_1) \int_{\tau=0}^{t_1} [b_1\, \mathrm{d}\sigma_{11}\, \mathrm{d}\sigma_{22} + b_3(\mathrm{d}\sigma_{11} + \mathrm{d}\sigma_{22})(\mathrm{d}\sigma_{11} + \mathrm{d}\sigma_{22} + \mathrm{d}\sigma_{33}) + b_5(\mathrm{d}\sigma_{11} + \mathrm{d}\sigma_{22} + \mathrm{d}\sigma_{33})^2 + b_6(\mathrm{d}\sigma_{11}^2 + \mathrm{d}\sigma_{22}^2 + \mathrm{d}\sigma_{33}^2)] \frac{1}{\|\mathrm{d}\bar{\sigma}\|} \tag{19}$$

and correspondingly for ϵ_{11}, ϵ_{33} and ϵ_{22}, ϵ_{33}. All other covariances are zero.

A SPECIAL CASE: POISSON PROCESS VISCOUS CREEP

Concrete creep may be taken as an example. In [3] the writer follows Benjamin *et al.*[1] who assume that there are essentially two independent stochastic creep components, viscous flow and delayed elasticity. The viscous flow part is modeled as being proportional to a nonhomogeneous Poisson process with mean rate being proportional to some decreasing function of time after loading. This Poisson process modeling of the viscous creep may as well be applicable for other materials than concrete. For simplicity of illustration we will concentrate on the viscous part herein and we will keep to the Poisson process assumption. The delayed elasticity part may be handled in a similar way using a Markov birth process as suggested in [1].

It should be admitted that these simplifying assumptions are subject to objections in the literature on concrete creep (numerous works of Bazant *et al.*). The intention of the present paper is, however, to illustrate a possibility of stochastic modeling of the creep phenomenon and to deduce some interesting consequences of the model. For this purpose the fundamental assumptions of the model are kept as simple as possible. For example, the Poisson process assumption makes it possible to express the covariance structure of the random strain tensor field solely in terms of a single scalar function $a(t, \tau)$ and the creep function, that is, the function that describes the mean deformation behavior. In a situation with scarcity or complete lack of interpretable experimental data such simple consistent modeling is particularly relevant.

Indeed, a remarkable property of the nonhomogeneous Poisson proceeds as a process of independent increments is that its variance equals its mean. This property is the key to express the functions $b_1, \ldots, b_8$ in terms of the function C and ν entering the mean creep equation, eqn (9). Of course, it is not too realistic to have the mean creep equal to the variance of the creep, Cinlar *et al.*[2]. However, this is corrected by using a creep process which is *proportional* to a nonhomogeneous Poisson process. Then the variance equals the proportionality factor times the mean. For uniaxial creep the proportionality factor is the aformentioned scalar function $a(t, \tau)$. To be specific, the modeling is as follows:

Corresponding to the principal stress increment state $\mathrm{d}\sigma_{11} = \alpha\, \mathrm{d}\sigma$, $\mathrm{d}\sigma_{22} = \beta\, \mathrm{d}\sigma$, $\mathrm{d}\sigma_{33} = \gamma\, \mathrm{d}\sigma$, $\mathrm{d}\sigma > 0$, in which α, β, γ are constants normalized such that $\alpha^2 + \beta^2 + \gamma^2 = 1$, and which is applied to time τ, we will assume that the strain tensor process for $t \geq \tau$ is proportional to a scalar nonhomogeneous Poisson process of intensity $\mu(t, \tau)\, \mathrm{d}\sigma$. With a and b being positive scalar functions of t and τ we will take the proportionality factors $\alpha a - (\beta + \gamma)b$, $\beta a - (\gamma + \alpha)b$, $\gamma a - (\alpha + \beta)b$ for ϵ_{11}, ϵ_{22}, and ϵ_{33} respectively and zero for all ϵ_{ij} for which $i \neq j$.

Referring to an infinitesimal volume element at the fixed place **r** the second moment properties of this model may be compared to the general second moment properties as specified by the mean equation, eqn (9), and the covariance equation, eqn (17), in which the Dirac delta function factor $\delta(\mathrm{r}_2 - \mathrm{r}_1)$ is removed by integration over the infinitesimal volume element. Writing the strain tensor at r to time t as ϵ_{ij} the Poisson model gives the mean

$$E[\epsilon_{11}] = (\alpha a - (\beta + \gamma)b)\mu\, \mathrm{d}\sigma \tag{20}$$

and the variance

$$\mathrm{Var}\,[\epsilon_{11}] = (\alpha a - (\beta + \gamma)b)^2 \mu\, \mathrm{d}\sigma. \tag{21}$$

For the mean, eqn (9) alternatively gives

$$E[\epsilon_{11}] = (\alpha - \nu(\beta + \gamma))C\, \mathrm{d}\sigma \tag{22}$$

while eqn (17) gives the variance

$$\mathrm{Var}[\epsilon_{11}] = [A_1\alpha^2 + A_2\alpha(\alpha+\beta+\gamma) + A_3(\alpha+\beta+\gamma)^2 + A_4(\alpha^2+\beta^2+\gamma^2)]\,\mathrm{d}\sigma \tag{23}$$

in which $A_1 = b_1 + 2b_2$, $A_2 = 2b_3 + 4b_4$, $A_3 = b_5 + 2b_7$, $A_4 = b_6 + 2b_8$.

Equations (20) and (22) are identical if and only if

$$\mu = \frac{1}{a}C \tag{24}$$

$$\left[\nu = \frac{b}{a}\right] \tag{25}$$

while eqns (21) and (23) are identical if and only if

$$A_1 = (1+\nu)^2 aC \tag{26}$$

$$A_2 = -2\nu(1+\nu)aC \tag{27}$$

$$A_3 = \nu^2 aC \tag{28}$$

$$A_4 = 0 \tag{29}$$

using eqns (24) and (25). Since $\mathrm{Var}[\epsilon_{ij}] = 0$ for $i \neq j$, it follows from eqn (18) that

$$b_2 = b_4 = b_7 = b_8 = 0. \tag{30}$$

Thus

$$b_1 = A_1,\ b_3 = \tfrac{1}{2}A_2,\ b_5 = A_3,\ b_6 = 0 \tag{31}$$

and the general eqn (16) becomes

$$\mathrm{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1), \epsilon_{kl}(\mathbf{r}_2, t_2)] = \delta(\mathbf{r}_2 - \mathbf{r}_1)\int_{\tau=0}^{t_1} [(1+\nu)^2\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\sigma_{kl} - \nu(1+\nu)(\delta_{ij}\,\mathrm{d}\sigma_{kl} + \delta_{kl}\,\mathrm{d}\sigma_{ij})\,\mathrm{d}\sigma_{ss} + \nu^2(\mathrm{d}\sigma_{ss})^2\delta_{ij}\delta_{kl}]\frac{aC}{\sqrt{(\mathrm{d}\sigma_{rs}\,\mathrm{d}\sigma_{rs})}} \tag{32}$$

valid for $t_1 \leq t_2$. In eqn (32), C, ν, a should be written as $C(t_1, \tau)$, $\nu(t_1, \tau)$, and $a(t_1, \tau)$ respectively while the stress tensor increment is taken at $(\mathbf{r}_1, \tau)$.

Compatibility and equilibrium

The strain tensor $\epsilon_{ij}(\mathbf{r}, t)$ must satisfy the compatibility equations

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0 \tag{33}$$

and the stress tensor increment $\mathrm{d}\sigma_{ij}(\mathbf{r}, t)$ must satisfy the equilibrium equations

$$\mathrm{d}\sigma_{ij,j}(\mathbf{r}, t) = 0 \tag{34}$$

assuming that there are no volume forces. Due to the stochastic constitutive equation, eqn (1), both eqns (33) and (34) are stochastic differential equations. Together with given time dependent boundary conditions, eqns (1), (33) and (34) determine a random strain field history and a corresponding random stress field history. The nonlinearity of eqn (1) leaves little hope, however, to find by known methods any exact solutions in terms of expectations and covariances. Thus a linearization of eqn (1) seems necessary.

Linearization of the stochastic constitutive equation

As in [3], the nonlinear constitutive equation, eqn (1), is avoided by changing it into the following constitutive equation, see Remark 1,

$$\epsilon_{ij}(\mathbf{r},t)=\int_{\tau=0}^{t}\int_{E[\tilde{\sigma}(\mathbf{r},\tau)]}^{E[\tilde{\sigma}(\mathbf{r},\tau)]+\mathrm{d}E[\tilde{\sigma}(\mathbf{r},\tau)]} S_{ijrs}(\mathbf{r},t,\tau,\tilde{u})\,\mathrm{d}u_{rs}+\int_{\tau=0}^{t}K_{ijrs}(t,\tau)(\mathrm{d}\sigma_{rs}(\mathbf{r},\tau)-\mathrm{d}E[\sigma_{rs}(\mathbf{r},\tau)]) \quad (35)$$

obtained from eqn (1) by linearization with respect to $\tilde{\sigma}$, $\tilde{\sigma}+\mathrm{d}\tilde{\sigma}$, and S_{ijrs}.

Remark 1.

The constitutive equation, eqn (1), is of the same type as a random variable W given by

$$W=\int_{Y}^{Y+Z}X(t)\,\mathrm{d}t \quad (36)$$

in which Y and Z are random variables while $X(t)$ is a random process. Assume that solely means and covariances are given for $(X(t),\ Y,\ Z)$. Then it is not possible to find mean and variance of W. It requires distributional information. However, by linearization in terms of the first order Taylor expansion of W at the mean $(E[X(t)],\ E[Y],\ E[Z])$, an approximate evaluation of $E[W]$ and $\mathrm{Var}[W]$ is possible. We have

$$W\overset{!}{=}\int_{E[Y]}^{E[Y]+E[Z]}E[X(t)]\,\mathrm{d}t+\int_{E[Y]}^{E[Y]+E[Z]}(X(t)-E[X(t)])\,\mathrm{d}t$$
$$-E[X(E[Y])](Y-E[Y])+E[X(E[Y]+E[Z])](Y+Z-E[Y]-E(Z)]) \quad (37)$$

which in case $E[X(t)]$ is a constant reduces to

$$W\overset{!}{=}\int_{E[Y]}^{E[Y]+E[Z]}X(t)\,\mathrm{d}t+E[X(t)](Z-E[Z]). \quad (38)$$

Equation (35) is of this last form.

The system of stochastic differential- and integral eqns (33), (34) and (35) is quite complicated to deal with in full generality. Therefore some special cases will be studied in the following.

SECOND MOMENT REPRESENTATION OF AVERAGE STRAIN IN BODY SUBJECTED TO GIVEN EXTERNAL FORCES

In case the material body $\mathscr{B}$ is subjected to given nonrandom external forces it follows from the equations of equilibrium that

$$\int_{\mathscr{B}}\mathrm{d}\sigma_{ij}(\mathbf{r},\tau)\quad\text{and}\quad\int_{\mathscr{B}}x_k\,\mathrm{d}\sigma_{ij}(\mathbf{r},\tau) \quad (39)$$

are both non random functions of τ. Therefore integration of eqn (35) with respect to $\mathbf{r}$ across $\mathscr{B}$ gives

$$\int_{\mathscr{B}}\epsilon_{ij}(\mathbf{r},t)=\int_{\tau=0}^{t}\int_{\mathscr{B}}\int_{E[\tilde{\sigma}(\mathbf{r},\tau)]}^{E[\tilde{\sigma}(\mathbf{r},\tau)]+\mathrm{d}E[\sigma(\mathbf{r},\tau)]}S_{ijrs}(\mathbf{r},t,\tau,\tilde{u})\,\mathrm{d}u_{rs} \quad (40)$$

while multiplication of eqn (35) by x_k and integration give

$$\int_{\mathscr{B}}x_k\epsilon_{ij}(\mathbf{r},t)=\int_{\tau=0}^{t}\int_{\mathscr{B}}x_k\int_{E[\tilde{\sigma}(\mathbf{r},\tau)]}^{E[\tilde{\sigma}(\mathbf{r},\tau)]+\mathrm{d}E[\tilde{\sigma}(\mathbf{r},\tau)]}S_{ijrs}(\mathbf{r},t,\tau,\tilde{u})\,\mathrm{d}u_{rs} \quad (41)$$

The expected stress tensor history $E[\tilde{\sigma}(\mathbf{r},\tau)]$ is the usual deterministic solution to the visco-elasticity problem as given by the constitutive equations, eqn (5), the compatibility equations, eqn (33), and the equilibrium equations, eqn (34).

The covariance properties of the left sides of eqns (40) and (41) may be calculated by standard technique using eqn (16). In particular, if the expected stress field is uniform throughout the body $\mathscr{B}$, the covariance between the average strain tensors $\epsilon_{ij}(t_1)$ and $\epsilon_{kl}(t_2)$ is

$$\begin{aligned}\operatorname{Cov}[\epsilon_{ij}(t_1), \epsilon_{kl}(t_2)] &= \frac{1}{(\operatorname{Vol}(\mathscr{B}))^2}\operatorname{Cov}\left[\int_{\mathscr{B}} \epsilon_{ij}(\mathbf{r}_1, t_1), \int_{\mathscr{B}} \epsilon_{kl}(\mathbf{r}_2, t_2)\right] \\ &= \frac{1}{(\operatorname{Vol}(\mathscr{B}))^2}\int_{\mathbf{r}_1\in\mathscr{B}}\int_{\mathbf{r}_2\in\mathscr{B}} \operatorname{Cov}[\epsilon_{ij}(\mathbf{r}_1, t_1), \epsilon_{kl}(\mathbf{r}_2, t_2)] \\ &= \frac{1}{\operatorname{Vol}(\mathscr{B})}\int_{\tau=0}^{t_1} (\text{integrand of eqn 16}) \qquad (42)\end{aligned}$$

in which $\operatorname{Vol}(\mathscr{B})$ is the volume of $\mathscr{B}$ and the stress tensor increments in the integrand are expected values.

Example 1.

Let $\mathscr{B}$ be a prismatic column of length L and cross sectional area A. The column is subjected to an axial force $N(\tau)$ varying in time and giving a uniform expected stress field throughout the column. It is

$$E[\sigma_{11}(\tau)] = \frac{N(\tau)}{A}, \quad E[\sigma_{ij}(\tau)] = 0 \quad \text{for} \quad (i, j) \neq (1, 1). \qquad (43)$$

The average axial strain $\epsilon_{11}(t)$ has the expectation, eqn (9),

$$E[\epsilon_{11}(t)] = \int_{\tau=0}^{t} C(t, \tau)\frac{\mathrm{d}N(\tau)}{A} \qquad (44)$$

while the covariance function is, eqn (42) and (16),

$$\operatorname{Cov}[\epsilon_{11}(t_1), \epsilon_{11}(t_2)] = \frac{1}{LA}\int_{\tau=0}^{t_1} (b_1 + 2b_2 + 2b_3 + 4b_4 + b_5 + b_6 + 2b_7 + 2b_8)(t_1, t_2, \tau)\frac{|\mathrm{d}N(\tau)|}{A} \qquad (45)$$

$$= \frac{1}{LA}\int_{\tau=0}^{t_1} a(t_1, \tau)C(t_1, \tau)\frac{|\mathrm{d}N(\tau)|}{A} \qquad (46)$$

for $t_1 \leq t_2$. The last expression corresponds to the particular Poisson process model that leads to eqn (32).

These results are seen to fit the corresponding results of the uniaxial model, [3: (2.8)–(2.11)]. The reader should also compare eqns (44) and (46) with [3: (2.12), (2.13)].

Example 2.

In case the column of Example 1 in place of the axial force $N(\tau)$ is subjected to a uniform hydrostatic pressure $p(\tau)$ we get, eqn (9),

$$E[\epsilon_{11}(t)] = \int_{\tau=0}^{t} (1 - 2\nu(t, \tau))C(t, \tau)\,\mathrm{d}p(\tau) \qquad (47)$$

and, eqns (42) and (16),

$$\operatorname{Cov}[\epsilon_{11}(t_1), \epsilon_{11}(t_2)] = \frac{1}{LA\sqrt{(3)}}\int_{\tau=0}^{t_1} (b_1 + 2b_2 + 6b_3 + 12b_4 + 9b_5 + 3b_6 + 18b_7 + 6b_8)(t_1, t_2, \tau)|\mathrm{d}p(\tau)| \qquad (48)$$

$$= \frac{1}{LA\sqrt{(3)}}\int_{\tau=0}^{t_1} (1 - 2\nu(t_1, \tau))^2 a(t_1, \tau)C(t_1, \tau)|\mathrm{d}p(\tau)| \qquad (49)$$

in which the last expression corresponds to eqn (32).

Example 3

Let the column of Example 1 be subjected solely to normal stresses acting on the cross-sections of the column ends such that the column is in a state of equilibrium. x_1 axis is in the axial direction while x_2 axis and x_3 axis are in the two principal directions of the cross-section. Origin is at the geometrical center of the column. Assume that the stresses varies linearly across the end cross-sections. Then the normal stress history may be written as

$$\sigma_{11}(\tau) = \frac{N(\tau)}{A} - \frac{M_3(\tau)}{I_3} x_2 - \frac{M_2(\tau)}{I_2} x_3 \tag{50}$$

in which $N(\tau)$ is axial force, $M_2(\tau)$ and $M_3(\tau)$ are bending moments with respect to x_2 axis and x_3 axis respectively, while I_2, I_3 are the moments of inertia with respect to x_2 axis, x_3 axis respectively.

Subjected to the stress history defined by eqn (50) and acting at each end of the column, a random strain field will develop in the body $\mathscr{B}$ of the column. This strain field will partly show up in terms of a random elongation and a random curvature of the column. Precise definitions of these terms, "random elongation" and "random curvature", are as follows. Define random processes $B(t)$, $C_2(t)$ $C_3(t)$ such that the integral

$$\int_{\mathscr{B}} (B(t) - C_3(t)x_2 - C_2(t)x_3 - \epsilon_{11}(\mathbf{r}, t))^2 \tag{51}$$

for each time t is as small as possible. From setting the partial derivatives of this integral with respect to B, C_2 and C_3 to zero respectively it is seen that

$$B(t) = \frac{1}{LA}\int_{\mathscr{B}} \epsilon_{11}(\mathbf{r}, t) \tag{52}$$

$$C_2(t) = -\frac{1}{LI_2}\int_{\mathscr{B}} x_3\epsilon_{11}(\mathbf{r}, t) \tag{53}$$

$$C_3(t) = -\frac{1}{LI_3}\int_{\mathscr{B}} x_2\epsilon_{11}(\mathbf{r}, t). \tag{54}$$

The random elongation of the column is defined as $LB(t)$, noting that $B(t)$ is simply the average strain $\epsilon_{11}(t)$ of $\mathscr{B}$. The random curvature with respect to x_2 axis is defined bo be $C_2(t)$, and with respect to x_3 axis to be $C_3(t)$.

It is seen that the mean and covariance properties of $C_2(t)$ and $C_3(t)$ may be calculated by use of eqn (41). For $C_2(t)$ we have

$$\begin{aligned} E[C_2(t)] &= -\frac{1}{I_2 L}\int_{\tau=0}^{t} C(t,\tau)\int_{\mathscr{B}} x_3\,\mathrm{d}\left(\frac{N(\tau)}{A} - \frac{M_3(\tau)}{I_3}x_2 - \frac{M_2(\tau)}{I_2}x_3\right) \\ &= \frac{1}{I_2}\int_{\tau=0}^{t} C(t,\tau)\,\mathrm{d}M_2(\tau) \end{aligned} \tag{55}$$

and

$$\mathrm{Cov}[C_2(t_1), C_2(t_2)] = \frac{1}{L}\int_{\tau=0}^{t} c(t_1, t_2, \tau)\int_{\omega}\left(\frac{x_3}{I_2}\right)^2 \left|\frac{1}{A}\mathrm{d}N(\tau) - \frac{x_2}{I_3}\mathrm{d}M_3(\tau) - \frac{x_3}{I_2}\mathrm{d}M_2(\tau)\right| \tag{56}$$

in which ω is the cross-section of the column while $c(t_1, t_2, \tau)$ is the function $(b_1 + 2b_2 + 2b_3 + 4b_4 + b_5 + b_6 + 2b_7 + 2b_8)(t_1, t_2, \tau)$ also appearing in eqn (45). Correspondance with unaxial theory is observed by comparison of eqns (55) and (56) with [3:(3.5), (3.8)].

Remark 2

The usefulness of uniaxial theory naturally depends on the possibility of taking into account variation of the internal forces along the x_1 axis. Let $\omega(x_1)$ be the cross-section at x_1. If it could be stated that

$$\int_{\omega(x_1)} \mathrm{d}\sigma_{ij}(\mathbf{r},\tau) \quad \text{and} \quad \int_{\omega(x_1)} x_k\,\mathrm{d}\sigma_{ij}(\mathbf{r},\tau) \tag{57}$$

are nonrandom functions of τ and x_1 determined solely by the nonrandom external forces acting on the column $\mathcal{B}$, then eqns (40) and (41) would be valid with integration across $\mathcal{B}$ changed to integration across $\omega(x_1)$ except for the approximation introduced by accepting eqn (35).

This statement is only true, however, for $i = j = 1$. For that case eqn (35) becomes

$$\epsilon_{11}(\mathbf{r},t) = \int_{\tau=0}^{t} \int_{E[\bar{\sigma}(\mathbf{r},\tau)]}^{E[\bar{\sigma}(\mathbf{r},\tau)]-\mathrm{d}E[\bar{\sigma}(\mathbf{r},\tau)]} S_{11rs}(\mathbf{r},t,\tau,\bar{u})\,\mathrm{d}u_{rs} + \int_{\tau=0}^{t} C(t,\tau)(\mathrm{d}\sigma_{11}(\mathbf{r},\tau) - \mathrm{d}E[\sigma_{11}(\mathbf{r},\tau)])$$
$$- \int_{\tau=0}^{t} \nu(t,\tau)C(t,\tau)(\mathrm{d}\sigma_{22}(\mathbf{r},\tau) + \mathrm{d}\sigma_{33}(\mathbf{r},\tau) - \mathrm{d}E[\sigma_{22}(\mathbf{r},\tau) + \sigma_{33}(\mathbf{r},\tau)]). \tag{58}$$

Integration across $\omega(x_1)$ gives the average axial strain at x_1:

$$\begin{aligned}\epsilon_{11}(x_1,t) &= \frac{1}{A}\int_{\omega(x_1)} \epsilon_{11}(\mathbf{r},t)\\ &= \frac{1}{A}\int_{\tau=0}^{t}\int_{\omega(x_1)}\int_{E[\bar{\sigma}(\mathbf{r},\tau)]}^{E[\bar{\sigma}(\mathbf{r},\tau)]+\mathrm{d}E[\bar{\sigma}(\mathbf{r},\tau)]} S_{11rs}(\mathbf{r},t,\tau,\bar{u})\,\mathrm{d}u_{rs}\\ &\quad - \int_{\tau=0}^{t} \nu(t,\tau)C(t,\tau)\frac{1}{A}\int_{\omega(x_1)} (\mathrm{d}\sigma_{22} + \mathrm{d}\sigma_{33} - \mathrm{d}E[\sigma_{22}+\sigma_{33}])(\mathbf{r},\tau).\end{aligned} \tag{59}$$

An analogous equation for $\int_{\omega(x_1)} x_k\epsilon_{11}(\mathbf{r},t)$ is obtained by writing x_k behind the two integral signs $\int_{\omega(x_1)}$ of eqn (59).

It is seen that the unaxial theory of bending given in [3: Section 3] comes out of the theory herein if the second term of eqn (59) may be neglected as compared to the first term. This is so if $\nu(t,\tau) \equiv 0$, of course, but also for sufficiently slender beam-columns as they are considered in the usual technical theory of elastic bending in which shear force deformations are neglected.

SUMMARY AND CONCLUSIONS

A triaxial constitutive tensor equation of strain as function of stress for stochastic viscous creep of a statistically isotropic material is formulated in terms of second moment white noise tensor fields. In the mean the usual constitutive equation of isotropic viscous creep appears. It contains two scalar functions of present time and of time of load application. These functions are the creep function and the Poisson ratio function.

With respect to covariances between two arbitrary strain tensor components at possibly different places and times the most general case is determined by eight scalar functions of present time and time of load application. For the special case of Poisson process viscous creep these eight functions are determined by only three functions of which the two are the creep function and the Poisson ratio function.

The stochastic nature of the constitutive tensor equation implies that also the local equilibrium equations and the compatibility equations are stochastic. For given boundary conditions it is only possible to obtain solutions if the constitutive tensor equation is linearized with respect to the random stress increments. Thus all solutions are approximations.

In this Part 1 of the paper, solutions are given in a simple way for the average strain in a body subjected to given external forces. For slender prismatic bodies the concept of random curvature and elongation is formulated within the triaxial theory. In this light, Part 1 terminates by a discussion of the uniaxial model of stochastic creep bending given by the writer in [3].

Part 2 of this paper concentrates on the special case of a biaxial model. Covariance function solutions are given for stress fields that are homogeneous in the mean.

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